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A Class of Univalent Harmonic Meromorphic Functions with Respect to k-Symmetric Points

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Abstract

After reading so many articles with respact to symmetric points such as [1], [6], [7], [8] in analytic functions, meromorphic functions, harmonic functions and harmonic meromorphic functions in punctured disk. In this article, I have an idea to introduce a class of univalent harmonic meromorphic functions with respact to k-symmetric ponts in outside of an unit disk. Some properties like coefficient condition, bounds and extreme points for functions belongs to the class has been studied.

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1 Introduction

A continuos function $f = u + iv$, is a complex-valued harmonic function in a domain D $\subset C$, if both u and v are real harmonic in D. Cluine and Sheil-small [2] investigated the family of all complex-valued harmonic mappings f defined on the open unit disk U, which admits the representation $f(z) = h(z) + \overline{g(z)}$ where h and g are analytic univalent in U.

Hengartner and Schober [3] considered the class $\sum_H$ of functions which are harmonic, meromorphic, orientation-preserving and univalent in $\widetilde{U} = \{z : |z| > 1\}$ so that $f(\infty) = \infty$. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)}, \tag{1}$$

where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}, \ g(z) = \sum_{n=1}^{\infty} b_n z^{-n} \tag{2}$$

are analytic in $\widetilde{U} = \{z = |z| > 1\}$.
Analogous to the concept given by Sakaguchi [5] for the class S_s^* of functions $f(z) \in S$, which are starlike with respect to symmetric points, the definition may be extended for harmonic meromorphic functions defined as follows:

Definition 1.1 *A function $f(z) \in \sum_H$ is said to be in the class $\sum_{HS}^*$ of starlike functions with respect to symmetric points, if it satisfies*

$$Re\left\{\frac{zf'(z)}{f_2(z)}\right\} > 0,$$

where $f_2(z) = \frac{1}{2}\left[f(z) - f(-z)\right]$.

Definition 1.2 *A function $f(z) \in \sum_H$ is said to be in the class $\sum_{H^k}$ of starlike functions with respect to symmetric points, if it satisfies*

$$Re\left\{\frac{zf'(z)}{f_k(z)}\right\} > 0,$$

where

$$f_k(z) = \frac{1}{k}\sum_{j=0}^{k-1} \epsilon^{-j} f(\epsilon^j z), \epsilon = \exp(\frac{2\pi i}{k}), \ k \geq 2.$$

Denote $\sum_{H^1} \equiv \sum_H^*, \ \sum_{H^2} \equiv \sum_{HS}^*$.

Definition 1.3 *A function $f(z) \in \sum_H$ is said to be in the class $\sum_{H^k}(\alpha)$, $(0 \leq \alpha < 1)$ of starlike functions with respect to symmetric points, if it satisfies*

$$Re\left\{\frac{zf'(z)}{f_k(z)}\right\} > \alpha,$$

where

$$f_k(z) = \frac{1}{k}\sum_{j=0}^{k-1} \epsilon^{-j} f(\epsilon^j z), \epsilon = \exp(\frac{2\pi i}{k}), k \geq 2,$$

which holds following relations

$$\begin{aligned} f_k(\epsilon^t z) &= \epsilon^t f_k(z), \\ f_k'(\epsilon^t z) &= \epsilon^t f_k'(z), \\ f_k''(\epsilon^t z) &= \epsilon^t f_k''(z). \end{aligned}$$

For $f = h + \overline{g}$, where h and g are of the form (2), Jahangiri [4] defined the modified Salagean operator of f as:

$$D^\lambda f(z) = D^\lambda h(z) + (-1)^\lambda \overline{D^\lambda g(z)};\ \lambda = 0, 1, 2...., \tag{3}$$

where

$$\begin{aligned} D^\lambda h(z) &= z^m + (-1)^\lambda \sum_{n=1}^{\infty} n^\lambda a_n z^{-n}, \\ D^\lambda g(z) &= (-1)^\lambda \sum_{n=1}^{\infty} n^\lambda b_n z^{-n}. \end{aligned}$$

Involving this operator D^λ, a class $\sum_{H^k}^{*}(\lambda, \alpha)$ is defined as follows.

Definition 1.4 *For $\lambda \in N_0$, $0 \le \alpha < 1$ and $k \ge 2$, let $\sum_{H^k}^{*}(\lambda, \alpha)$ denote the class of multivalent meromorphic harmonic functions f of the form (2) satisfying*

$$Re\left\{\frac{D^{\lambda+1} f(z)}{D^\lambda f_k(z)}\right\} > \alpha, \tag{4}$$

where

$$f_k(z) = h_k(z) + \overline{g_k(z)},$$

$$h_k(z) = z + \sum_{n=1}^{\infty} a_n \psi_n z^{-n} \quad g_k(z) = \sum_{n=1}^{\infty} b_n \psi_n z^{-n}, \tag{5}$$

$$\begin{aligned} \psi_n &= \frac{1}{k} \sum_{j=0}^{k-1} \epsilon^{-(n+1)j}, \quad \left(k \ge 2;\ \epsilon = \exp(\frac{2\pi i}{k})\right) \\ &= \begin{cases} 1, & n+1 = lk,\ l \in N, \\ 0, & n+1 = lk+1,\ l \in N \end{cases} \end{aligned} \tag{6}$$

and

$$D^\lambda f_k(z) = D^\lambda h_k(z) + (-1)^\lambda \overline{D^\lambda g_k(z)}.$$

Let $\sum_{\overline{H^k}}^{*}(\lambda, \alpha)$ denote the subclass of $\sum_{H^k}^{*}(\lambda, \alpha)$ consisting of functions of the form $f_\lambda = h_\lambda + \overline{g_\lambda}$ such that

$$h_\lambda(z) = z + (-1)^\lambda \sum_{n=1}^{\infty} |a_n| z^{-n},\ g_\lambda(z) = -\sum_{n=1}^{\infty} |b_n| z^{-n}. \tag{7}$$

Also, let $f_{k_\lambda} = h_{k_\lambda} + \overline{g_{k_\lambda}}$ where h_{k_λ} and g_{k_λ} are of the form

$$h_{k_\lambda}(z) = z + (-1)^\lambda \sum_{n=1}^{\infty} |a_n| \psi_n z^{-n}, g_{k_\lambda}(z) = -\sum_{n=1}^{\infty} |b_n| \psi_n z^{-n}, \tag{8}$$

where ψ_n is given by (6).

In this article, a result is based on the class $\sum_{H^k}^*(\lambda,\alpha)$ is obtained and a sufficient coefficient condition for functions $f = h + \overline{g}$, where h and g are of the form (2) to be in the class $\sum_{H^k}^*(\lambda,\alpha)$ is determined. It is shown that this coefficient condition is also necessary for functions to be in its subclass $\sum_{\overline{H^k}}^*(\lambda,\alpha)$. Furthermore, bounds and extreme points for functions in $\sum_{\overline{H^k}}^*(\lambda,\alpha)$ class are obtained.

2 A Result for class $\sum_{H^k}^*(\lambda,\alpha)$

In this section, a result for the class $\sum_{H^k}^*(\lambda,\alpha)$ is derived.

Theorem 2.1 *For $\lambda \in N_0$, $0 \le \alpha < 1$ and $k \ge 2$, if $f \in \sum_{H^k}^*(\lambda,\alpha)$, then $D^\lambda f_k(z) \in \sum_{H^k}^*(\lambda,\alpha)$.*

Proof If $f \in \sum_{H^k}^*(\lambda,\alpha)$, then from the Definition 1.4, it follows that

$$Re\left\{\frac{D^{\lambda+1}f(z)}{D^\lambda f_k(z)}\right\} > \alpha, z \in \widetilde{U}.$$

Hence

$$Re\left\{\frac{D^{\lambda+1}f(\epsilon^\mu z)}{D^\lambda f_k(\epsilon^\mu z)}\right\} > \alpha,\ \epsilon^\mu z \in \widetilde{U},\ \epsilon = \exp(\frac{2\pi i}{k}),\ \mu = 0, 1..(k-1).$$

From the Definition 1.4, it follows that $D^\lambda f_k(\epsilon^\mu z) = \epsilon^\mu D^\lambda f_k(z)$, $\mu = 0, 1..(k-1)$. Thus

$$\frac{1}{k}\sum_{\mu=0}^{k-1} Re\left\{\frac{D^{\lambda+1}f(\epsilon^\mu z)}{D^\lambda f_k(\epsilon^\mu z)}\right\} > \alpha.$$

Since

$$D^{\lambda+1}f_k(\epsilon^\mu z) = \frac{1}{k}\sum_{j=0}^{k-1}\epsilon^{-j}D^{\lambda+1}f(\epsilon^j z),$$

then

$$Re\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\frac{D^{\lambda+1}f(\epsilon^\mu z)}{\epsilon^\mu D^\lambda f_k(z)}\right\} > \alpha$$

or

$$Re\left\{\frac{D^{\lambda+1}f_k(z)}{D^\lambda f_k(z)}\right\} > \alpha,$$

which proves the result.

3 Coefficient Conditions

In this section, sufficient coefficient condition for a function $f \in \sum_H$ to be in $\sum^*_{H^k}(\lambda,\alpha)$ is derived and then it is shown that this coefficient condition is necessary for its subclass $\sum^*_{\overline{H^k}}(\lambda,\alpha)$.

Theorem 3.1 *Let $f = h + \overline{g}$, where h and g are given by (2) and $f_k = h_k + \overline{g_k}$, where h_k and g_k are given by (5) satisfying*

$$\sum_{n=1}^{\infty} n^{\lambda}\left[\frac{(n+\alpha\psi_n)}{(1-\alpha)}|a_n| + \frac{(n-\alpha\psi_n)}{(1-\alpha)}|b_n|\right] \leq 1, \tag{9}$$

or, equivalently

$$\begin{aligned}&\sum_{lk=2,k\geq 1}^{\infty}\left[\frac{lk-(1-\alpha)}{(1-\alpha)}\right](lk-1)^{\lambda}|a_{lk-1}| \\ &+\sum_{lk=2,k\geq 1}^{\infty}\left[\frac{lk-(1+\alpha)}{(1-\alpha)}\right](lk-1)^{\lambda}|b_{lk-1}| \\ &+\sum_{lk=2,k\geq 2}^{\infty}\left(\frac{lk}{(1-\alpha)}\right)(|a_{lk}|+|b_{lk}|)\left(\frac{lk}{(1-\alpha)}\right)^{\lambda} \leq 1,\end{aligned}$$

where $\lambda \in N_0$, $0 \leq \alpha < 1$, $k \in N$, then f is harmonic, orientation-preserving in $\widetilde{U}$ and $f \in \sum^*_{H^k}(\lambda,\alpha)$.

Proof To show that f is orientation-preserving in $\widetilde{U}$, it only needs to show that $|h'(z)| \geq |g'(z)|$ in $\widetilde{U}$. Thus

$$\begin{aligned}|h'(z)| &= \left|1-\sum_{n=1}^{\infty} n a_n z^{-(n+1)}\right| \\ &\geq 1-\sum_{n=1}^{\infty} n|a_n||z|^{-(n+1)} \\ &= 1-\sum_{n=1}^{\infty} n|a_n| r^{-(n+1)} \\ &\geq 1-\sum_{n=1}^{\infty} n|a_n| \\ &\geq 1-\sum_{n=1}^{\infty} n^{\lambda}\frac{(n+\alpha\psi_n)}{(1-\alpha)}|a_n| \\ &\geq \sum_{n=1}^{\infty} n^{\lambda}\frac{(n-\alpha\psi_n)}{(1-\alpha)}|b_n| \geq \sum_{n=1}^{\infty} n|b_n| \\ &\geq \sum_{n=1}^{\infty} n|b_n| r^{-(n+1)} \geq \sum_{n=1}^{\infty} n|b_n||z|^{-(n+1)} \geq |g'(z)|.\end{aligned}$$

Now, in order to show $f \in \sum_{H^k}^{*}(\lambda, \alpha)$, it suffices to show that

$$Re\left\{\frac{D^{\lambda+1}f(z)}{D^{\lambda}f_k(z)}\right\} > \alpha \tag{10}$$

or,

$$Re\left\{\frac{D^{\lambda+1}h(z) - (-1)^{\lambda}\overline{D^{\lambda+1}g(z)}}{D^{\lambda}h_k(z) + (-1)^{\lambda}\overline{D^{\lambda}g_k(z)}}\right\} > \alpha,$$

where $z = re^{i\theta}$, $0 < r \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \alpha < 1$.

Let

$$A(z) := D^{\lambda+1}h(z) - (-1)^{\lambda}\overline{D^{\lambda+1}g(z)} \tag{11}$$

and

$$B(z) := D^{\lambda}h_k(z) + (-1)^{\lambda}\overline{D^{\lambda}g_k(z)}. \tag{12}$$

It is observed that (10) holds if

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \ge 0. \tag{13}$$

From (11) and (12), it follows that

$$\begin{aligned}
&|A(z) + (1-\alpha)B(z)| \\
&= \left|D^{\lambda+1}h(z) - (-1)^{\lambda}\overline{D^{\lambda+1}g(z)} + (1-\alpha)\left(D^{\lambda}h_k(z) + (-1)^{\lambda}\overline{D^{\lambda}g_k(z)}\right)\right| \\
&= \left|(2-\alpha)z - (-1)^{\lambda}\sum_{n=1}^{\infty}\left[n-(1-\alpha)\psi_n\right]n^{\lambda}a_n z^{-n} + \sum_{n=1}^{\infty}\left[n+(1-\alpha)\psi_n\right]n^{\lambda}\overline{b_n z^{-n}}\right| \\
&\ge (2-\alpha)\,|z| - \sum_{n=1}^{\infty}\left[n-(1-\alpha)\psi_n\right]n^{\lambda}\,|a_n|\,|z|^{-n} + \sum_{n=1}^{\infty}\left[n+(1-\alpha)\psi_n\right]n^{\lambda}\,|b_n|\,|z|^{-n}
\end{aligned}$$

and

$$\begin{aligned}
&|A(z) - (1+\alpha)B(z)| \\
&= \left|D^{\lambda+1}h(z) - (-1)^{\lambda}\overline{D^{\lambda+1}g(z)} - (1+\alpha)\left(D^{\lambda}h_k(z) + (-1)^{\lambda}\overline{D^{\lambda}g_k(z)}\right)\right| \\
&= \left|(-\alpha)z - (-1)^{\lambda}\sum_{n=1}^{\infty}\left[n+(1+\alpha)\psi_n\right]n^{\lambda}a_n z^{-n} + \sum_{n=1}^{\infty}\left[n-(1+\alpha)\psi_n\right]n^{\lambda}\overline{b_n z^{-n}}\right| \\
&= \left|\alpha z + (-1)^{\lambda}\sum_{n=1}^{\infty}\left[n+(1+\alpha)\psi_n\right]n^{\lambda}a_n z^{-n} - \sum_{n=1}^{\infty}\left[n-(1+\alpha)\psi_n\right]n^{\lambda}\overline{b_n z^{-n}}\right| \\
&\le \alpha z + \sum_{n=1}^{\infty}\left[n+(1+\alpha)\psi_n\right]n^{\lambda}\,|a_n|\,|z|^{-n} - \sum_{n=1}^{\infty}\left[n-(1+\alpha)\psi_n\right]n^{\lambda}\,|b_n|\,z^{-n}
\end{aligned}$$

Thus

$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)|$

$$
\begin{aligned}
&\geq \;\; 2\left(1-\alpha\right)\left|z\right|-2\sum_{n=1}^{\infty}\left[n+\alpha\psi_{n}\right]n^{\lambda}\left|a_{n}\right|\left|z\right|^{-n} \\
&\qquad -2\sum_{n=1}^{\infty}\left[n-\alpha\psi_{n}\right]n^{\lambda}\left|b_{n}\right|\left|z\right|^{-n} \\
&\geq \;\; 2\left|z\right|\left\{\left(1-\alpha\right)-\sum_{n=1}^{\infty}\left[n+\alpha\psi_{n}\right]n^{\lambda}\left|a_{n}\right|\left|z\right|^{-(n+1)}\right. \\
&\qquad \left.-\sum_{n=1}^{\infty}\left[n-\alpha\psi_{n}\right]n^{\lambda}\left|b_{n}\right|\left|z\right|^{-(n+1)}\right\} \\
&\geq \;\; 2\left|z\right|\left(1-\alpha\right)\left\{1-\sum_{n=1}^{\infty}\left[\frac{n+\alpha\psi_{n}}{(1-\alpha)}\right]n^{\lambda}\left|a_{n}\right|\left|z\right|^{-(n+1)}\right. \\
&\qquad \left.-\sum_{n=1}^{\infty}\left[\frac{n-\alpha\psi_{n}}{(1-\alpha)}\right]n^{\lambda}\left|b_{n}\right|\left|z\right|^{-(n+1)}\right\} \\
&\geq \;\; 2\left(1-\alpha\right)\left\{1-\sum_{n=1}^{\infty}\left[\frac{n+\alpha\psi_{n}}{(1-\alpha)}\right]n^{\lambda}\left|a_{n}\right|\right. \\
&\qquad \left.\sum_{n=1}^{\infty}\left[\frac{n-\alpha\psi_{n}}{(1-\alpha)}\right]n^{\lambda}\left|b_{n}\right|\right\}
\end{aligned} \tag{14}
$$

From the definition of ψ_n, it follows that

$$
\psi_{n}=\begin{cases} 1, & n+1=lk, l\in N,\ k\geq 1 \\ 0, & n+1=lk+1,\ l\in N\ ,\ k\geq 2. \end{cases} \tag{15}
$$

Substituting (15) in (14), then (14) is equivalent to

$$
\begin{aligned}
&\left|A(z)+(1-\alpha)B(z)\right|-\left|A(z)-(1+\alpha)B(z)\right| \\
&\geq 2\left(1-\alpha\right)\left\{1-\sum_{lk=2,k\geq 1}^{\infty}\left[\frac{lk-(1-\alpha)}{(1-\alpha)}\right]\left(lk-1\right)^{\lambda}\left|a_{lk-1}\right|\right. \\
&-\sum_{lk=2,k\geq 1}^{\infty}\left[\frac{lk-(1+\alpha)}{(1-\alpha)}\right]\left(\frac{lk-m}{m}\right)^{\lambda}\left|b_{lk-m}\right| \\
&\left.-\sum_{lk=2,k\geq 2}^{\infty}\left(\frac{lk}{(1-\alpha)}\right)\left(\left|a_{lk}\right|+\left|b_{lk}\right|\right)\left(lk\right)^{\lambda}\right\} \\
&\geq 0.\ by(13).
\end{aligned}
$$

Thus, this completes the proof of the Theorem.

Theorem 3.2 *Let $f_\lambda = h_\lambda + \overline{g_\lambda}$, where h_λ and g_λ are of the form (7), and $f_{k_\lambda} = h_{k_\lambda} + \overline{g_{k_\lambda}}$, where h_{k_λ} and g_{k_λ} are of the form (8). Then, $f_\lambda \in \sum^{*}_{\overline{H^k}}(\lambda,\alpha)$, if and only if inequality (9) holds for the coefficient of $f_\lambda = h_\lambda + \overline{g_\lambda}$ and $f_{k_\lambda} = h_{k_\lambda} + \overline{g_{k_\lambda}}$.*

Proof Since $\sum_{\overline{H^k}}^{*}(\lambda,\alpha) \subset \sum_{H^k}^{*}(\lambda,\alpha)$, if part is proved in Theorem 3.1. It only needs to prove the "only if" part of the Theorem. For this, it suffices to show that $f_\lambda \notin \sum_{\overline{H^k}}^{*}(\lambda,\alpha)$ if the condition (9) does not hold. If $f_\lambda \in \sum_{\overline{H^k}}^{*}(\lambda,\alpha)$, then writing corresponding series expansions in (4) , it follows that $Re\left\{\frac{\xi(z)}{\eta(z)}\right\} \geq 0$ for all values of z in $\widetilde{U}$, where

$$\begin{aligned} \xi(z) &= D^{\lambda+1}h_\lambda(z) - (-1)^\lambda\overline{D^{\lambda+1}g_\lambda(z)} - \alpha D^\lambda h_{k_\lambda}(z) - \alpha(-1)^\lambda\overline{D^\lambda g_{k_\lambda}(z)} \\ &= z - \sum_{n=1}^{\infty}\left[\frac{n+\alpha\psi_n}{(1-\alpha)}\right](n)^\lambda |a_n| z^{-n} - \sum_{n=1}^{\infty}\left[\frac{n-\alpha\psi_n}{(1-\alpha)}\right] n^\lambda |b_n| |z|^{-n} \end{aligned}$$

and

$$\begin{aligned} \eta(z) &= D^\lambda h_{k_\lambda}(z) + (-1)^\lambda\overline{D^\lambda g_{k_\lambda}(z)} \\ &= z + \sum_{n=1}^{\infty} n^\lambda \psi_n \left[|a_n| z^{-n} + |b_n| \overline{z}^{-n}\right]. \end{aligned}$$

Since

$$\left|\frac{\xi(z)}{\eta(z)}\right| \geq Re\left\{\frac{\xi(z)}{\eta(z)}\right\} \geq 0,$$

hence the condition $Re\left\{\frac{\xi(z)}{\eta(z)}\right\} \geq 0$ is equivalent to

$$\frac{1-\sum_{n=1}^{\infty} n^\lambda\left[\left[\frac{n+\alpha\psi_n}{(1-\alpha)}\right]|a_n| + \left[\frac{n-\alpha\psi_n}{(1-\alpha)}\right]|b_n|\right] r^{-(n+1)}}{1+\sum_{n=1}^{\infty} n^\lambda\psi_n\left[|a_n|+|b_n|\right] r^{-(n+1)}} \geq 0. \tag{16}$$

Now if the condition (9) does not holds then the numerator of (16) is non-positive for r sufficiently close to 1, which contradicts that $f_\lambda \in \sum_{\overline{H^k}}^{*}(\lambda,\alpha)$ and this proves the required result.
Taking $\lambda = 0$, in Theorems 3.1 and 3.2, following results are obtained.

Corollary 3.3 *Let $f = h + \overline{g}$, where h and g are given by (2) and $f_k = h_k + \overline{g_k}$, where h_k and g_k are given by (5) satisfying*

$$\sum_{n=1}^{\infty}\left[\frac{(n+\alpha\psi_n)}{(1-\alpha)}|a_n| + \frac{(n-\alpha\psi_n)}{(1-\alpha)}|b_n|\right] \leq 1,$$

or, equivalently

$$\begin{aligned} &\sum_{lk=2,k\geq 1}^{\infty}\left[\frac{lk-(1-\alpha)}{(1-\alpha)}\right]|a_{lk-1}| + \sum_{lk=2,k\geq 1}^{\infty}\left[\frac{lk-(1+\alpha)}{(1-\alpha)}\right]|b_{lk-1}| \\ &+ \sum_{lk=2,k\geq 2}^{\infty}\left(\frac{lk}{(1-\alpha)}\right)(|a_{lk}|+|b_{lk}|) \leq 1, \end{aligned}$$

where $0 \leq \alpha < 1$, $k \in N$, then f is harmonic, orientation-preserving in $\widetilde{U}$ and $f \in \sum_{H^k}(\alpha)$. Furthermore, $f \in \sum_{\overline{H^k}}(\alpha)$, if and only if above inequality holds.

4 Bounds and Extreme Points

In this section, bounds and extreme points for functions belonging to the class $\sum_{\overline{H^k}}^{*}(\lambda,\alpha)$ are estimated.

Theorem 4.1 *If $f_\lambda = h_\lambda + \overline{g_\lambda} \in \sum_{\overline{H^k}}^{*}(\lambda,\alpha)$ for $0 \le \alpha < 1$, $0 < |z| = r < 1$, then*

$$r - \frac{1}{r} \le |f_\lambda(z)| \le r + \frac{1}{r}. \tag{17}$$

Proof Let $f_\lambda = h_\lambda + \overline{g_\lambda} \in \sum_{\overline{H^k}}^{*}(\lambda,\alpha)$. Taking the absolute value of f_λ it follows that

$$\begin{aligned}
|f_\lambda(z)| &= \left| z + (-1)^\lambda \sum_{n=1}^{\infty} |a_n| z^{-n} - \overline{\sum_{n=1}^{\infty} |b_n| z^{-n}} \right| \\
&\le r + \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^{-n} \\
&\le r + r^{-1} \sum_{n=1}^{\infty} (|a_n| + |b_n|) \\
&\le r + r^{-1} \sum_{n=1}^{\infty} \frac{(1-\alpha)}{n-\alpha\psi_n} \frac{n-\alpha\psi_n}{(1-\alpha)} n^\lambda \left(\frac{1}{n}\right)^\lambda (|a_n| + |b_n|) \\
&\le r + r^{-1} \frac{(1-\alpha)}{(1-\alpha)} \sum_{n=1}^{\infty} n^\lambda \frac{n-\alpha\psi_n}{(1-\alpha)} (|a_n| + |b_n|) \\
&\le r + r^{-1} \sum_{n=1}^{\infty} n^\lambda \left[\frac{n+\alpha\psi_n}{(1-\alpha)} |a_n| + \frac{n-\alpha\psi_n}{(1-\alpha)} |b_n| \right] \\
&\le r + r^{-1}
\end{aligned}$$

and

$$\begin{aligned}
|f_\lambda(z)| &= \left| z + (-1)^\lambda \sum_{n=1}^{\infty} |a_n| z^{-n} - \overline{\sum_{n=1}^{\infty} |b_n| z^{-n}} \right| \\
&\ge r - \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^{-n} \\
&\ge r - r^{-1} \sum_{n=1}^{\infty} (|a_n| + |b_n|) \\
&\ge r - r^{-1} \sum_{n=1}^{\infty} \frac{(1-\alpha)}{n-\alpha\psi_n} \frac{n-\alpha\psi_n}{(1-\alpha)} n^\lambda \left(\frac{1}{n}\right)^\lambda (|a_n| + |b_n|) \\
&\ge r - r^{-1} \frac{(1-\alpha)}{(1-\alpha)} \sum_{n=1}^{\infty} n^\lambda \frac{n-\alpha\psi_n}{(1-\alpha)} (|a_n| + |b_n|) \\
&\ge r - r^{-1} \sum_{n=1}^{\infty} n^\lambda \left[\frac{n+\alpha\psi_n}{(1-\alpha)} |a_n| + \frac{n-\alpha\psi_n}{(1-\alpha)} |b_n| \right] \\
&\ge r - r^{-1}.
\end{aligned}$$

This proves the required result.
The bounds given in Theorem 4.1 holds for the functions $f_\lambda = h_\lambda + \overline{g_\lambda}$, and it also found that these bounds also holds for functions $f = h + \overline{g}$.

Theorem 4.2 *Let $f_\lambda = h_\lambda + \overline{g_\lambda}$, where h_λ and g_λ are of the form (7) then $f_\lambda \in \sum_{\overline{H^k}}^*(\lambda,\alpha)$, if and only if f_λ can be expressed as:*

$$f_\lambda(z) = \sum_{n=0}^{\infty} \left(x_n h_{\lambda_n}(z) + y_n g_{\lambda_n}(z) \right), \tag{18}$$

where $z \in \widetilde{U}$ and

$$h_{\lambda_0}(z) = z, \quad h_{\lambda_n}(z) = z + (-1)^\lambda \frac{(1-\alpha)}{[n+\alpha\psi_n]\, n^\lambda} z^{-n}, \tag{19}$$

$$g_{\lambda_0}(z) = z, \quad g_{\lambda_n}(z) = z - \frac{(1-\alpha)}{[n-\alpha\psi_n]\, n^\lambda} \overline{z^{-n}} \tag{20}$$

for $n = 1, 2, 3, \ldots\ldots$, and

$$\sum_{n=0}^{\infty} (x_n + y_n) = 1, \qquad x_n,\ y_n \geq 0. \tag{21}$$

In particular, the extreme points of $\sum_{\overline{H^k}}^*(\lambda,\alpha)$ are h_{λ_n} and g_{λ_n}.

ProofLet

$$\begin{aligned}
f_\lambda(z) &= \sum_{n=0}^{\infty} \left(x_n h_{\lambda_n}(z) + y_n g_{\lambda_n}(z) \right) \\
&= x_0 h_{\lambda_0}(z) + y_0 g_{\lambda_0}(z) + \sum_{n=1}^{\infty} x_n \left(z + (-1)^\lambda \frac{(1-\alpha)}{[n+\alpha\psi_n]\, n^\lambda} z^{-n} \right) \\
&\qquad + \sum_{n=1}^{\infty} y_n \left(z - \frac{(1-\alpha)}{[n-\alpha\psi_n]\, n^\lambda} \overline{z^{-n}} \right) \\
&= \sum_{n=0}^{\infty} (x_n + y_n)\, z + \sum_{n=1}^{\infty} \left\{ (-1)^\lambda \left(\frac{(1-\alpha)}{[n+\alpha\psi_n]\, n^\lambda} \right) x_n\, z^{-n} \right. \\
&\qquad \left. - \frac{(1-\alpha)}{[n-\alpha\psi_n]\, n^\lambda} y_n \overline{z^{-n}} \right\}.
\end{aligned}$$

Thus by Theorem 3.2, it is noted that $f_\lambda \in \sum_{\overline{H^k}}^*(\lambda,\alpha)$, since,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^\lambda \left\{ \frac{[n+\alpha\psi_n]}{(1-\alpha)} \left(\frac{(1-\alpha)}{[n+\alpha\psi_n]\, n^\lambda} x_n \right) \right. \\
&\quad \left. + \frac{[n-\alpha\psi_n]}{(1-\alpha)} \left(\frac{(1-\alpha)}{[n-\alpha\psi_n]\, n^\lambda} y_n \right) \right\} \\
&= \sum_{n=1}^{\infty} (x_n + y_n) = (1 - x_0 - y_0) \leq 1.
\end{aligned}$$

Conversely, suppose that $f_\lambda \in \sum_{\overline{H^k}}^*(\lambda, \alpha)$, then (9) holds.
Setting

$$\begin{aligned} x_n &= \frac{[n+\alpha\psi_n]}{(1-\alpha)} n^\lambda |a_n|, \\ y_n &= \frac{[n-\alpha\psi_n]}{(1-\alpha)} n^\lambda |b_n|, \end{aligned}$$

which satisfy (14), thus

$$\begin{aligned} f_\lambda(z) &= z + (-1)^\lambda \sum_{n=1}^{\infty} |a_n| z^{-n} - \overline{\sum_{n=1}^{\infty} |b_n| z^{-n}} \\ &= z + (-1)^\lambda \sum_{n=1}^{\infty} \frac{(1-\alpha)}{[n+\alpha\psi_n]\, n} x_n z^{-n} - \sum_{n=1}^{\infty} \frac{(1-\alpha)}{[n-\alpha\psi_n]\, n^\lambda} y_n \overline{z}^{-n} \\ &= z + \sum_{n=1}^{\infty} [h_{\lambda_n} - z]\, x_n + \sum_{n=1}^{\infty} [g_{\lambda_n} - z]\, y_n \\ &= z\left[1 - \sum_{n=1}^{\infty} x_n - \sum_{n=1}^{\infty} y_n\right] + \sum_{n=1}^{\infty} h_{\lambda_n} x_n + \sum_{n=1}^{\infty} g_{\lambda_n} y_n \\ &= x_0 h_{\lambda_0} + y_0 g_{\lambda_0} + \sum_{n=1}^{\infty} h_{\lambda_n} x_n + \sum_{n=1}^{\infty} g_{\lambda_n} y_n \\ &= \sum_{n=0}^{\infty} \left(x_n h_{\lambda_n}(z) + y_n g_{\lambda_n}(z)\right). \end{aligned}$$

This proves the Theorem.

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